MATHEMATICAL ECONOMICS
Exam 2, 16.11.2014
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Only calculators provided by the university are allowed!

1. Assume following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}.$$

   Calculate the following:

   a) sum $A + B$
   b) transpose B^T
   c) product AB
   d) determinant $|A|$
   e) inverse A^{-1}
   f) rank of matrix B

2. a) Solve equation $2x^{-\frac{1}{2}} = 1$.
   b) Differentiate function $f(x) = (x^2 - x + 2)e^{2x}$.
   c) Compute definite integral $\int_0^1 (x^2 + 1)dx$.

3. Equation $x^2 + z^2 + 3zy + 2yx + y^2 = 11$ holds when $x = 0$, $y = 2$ and $z = 1$. Does the equation determine x as a function of y and z around point $(y, z) = (2, 1)$? If yes, calculate the partial derivatives $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial z}$ at point $(y, z) = (2, 1)$.

4. Solve the utility maximization problem

$$\max_{\{c_1, c_2\}} \{\alpha \ln c_1 + \beta \ln c_2\} \quad \text{s.t.} \quad p_1 c_1 + p_2 c_2 \leq w$$

   for the optimal consumption bundle $(c_1^*, c_2^*) \in \mathbb{R}^2_{++}$. Parameters α, β, p_1, p_2 and w are all strictly positive.

5. a) Find an explicit solution for difference equation $y_{t+1} = \frac{1}{2}y_t + 5$ with initial condition $y_0 = 10$.
   b) Find stationary points of differential equation $y'(t) = y(t)^3 - y(t)$ and use graphical analysis to determine whether the stationary points are locally stable or not. Sketch the solution function $y(t)$ assuming initial condition $y(0) = \frac{1}{2}$.

Following lemmas, which consider testing for local quality of critical points, may or may not be useful:

Lemma 1 (definiteness of Hessian) *Matrix A is*

1. *positive definite, if* $|A_i| > 0 \quad \forall i = 1, \ldots, n$.
2. *negative definite, if* $|A_1| < 0, |A_2| > 0, |A_3| < 0$, *etc. (sign alternates with first sign negative).*
3. *positive semi-definite, if* $|A_i| \geq 0 \quad \forall i = 1, \ldots, n$.
4. *negative semi-definite, if* $|A_1| \leq 0, |A_2| \geq 0, |A_3| \leq 0$, *etc.*
5. *indefinite, if none of the above holds.*

Lemma 2 (using bordered Hessian) *Assume an equality-constrained optimization problem with n choice variables and m constraints. Critical points of the Lagrangian can be classified by examining subdeterminants of the bordered Hessian.*

A critical point is a strict local minimum if the signs of the $n - m$ largest subdeterminants are all equal to the sign of $(-1)^m$. A critical point is a strict local maximum if the signs of $n - m$ largest subdeterminants alternate and the sign of the largest subdeterminant is equal to the sign of $(-1)^n$.